

# Landau's Problem of Degenerate Plasma Oscillations in Slab with Specular Boundary Conditions

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In the present paper the linearized problem of plasma oscillations in slab (particularly, thin films) in external longitudinal alternating electric field is solved analytically. Specular boundary conditions of electron reflection from the plasma boundary are considered. Coefficients of continuous and discrete spectra of the problem are found, and electron distribution function on the plasma boundary and electric field are expressed in explicit form. Absorption of energy of electric field in slab is calculated.

Refs. 34. Figs. 2.

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## 1. INTRODUCTION

The present paper is devoted to degenerate electron plasma behaviour research. Analysis of processes taking place in plasma under effect of external electric field, plasma waves oscillations with various types of conditions of electron reflection from the boundary has important significance today in connection with problems of such intensively developing fields as microelectronics and nanotechnologies [1] – [6].

The concept of "plasma" appeared in the works of Tonks and Langmuir for the first time (see [7]–[9]), the concept of "plasma frequency" was

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introduced in the same works and first questions of plasma oscillations were considered there. However, in these works equation for the electric field was considered separately from the kinetic equation.

A.A. Vlasov [10] for the first time introduced the concept of "self-consistent electric field" and added the corresponding item to the kinetic equation. Now equations describing plasma behaviour consist of anchor system of equations of Maxwell and Boltzmann. The problem of electron plasma oscillations was considered by A.A. Vlasov [10] by means of solution of the kinetic equation which included self-consistent electric field.

L.D. Landau [11] had supposed that outside of the half-space containing degenerate plasma external electromagnetic field causing oscillations in plasma is situated. By this Landau has formulated a boundary condition on the plasma boundary. After that the problem of plasma oscillation turns out to be formulated correctly as a boundary problem of mathematical physics.

In [11] L.D. Landau has solved analytically by Fourier series the problem of collisionless plasma behaviour in a half-space, situated in external longitudinal (perpendicular to the surface) electric field, in conditions of specular reflection of electrons from the boundary.

Further the problem of electron plasma oscillations was considered by many authors. Full analytical solution of the problem is given in the works [12] and [13].

This problem has important significance in the theory of plasma (see, for instance, [2], [14] and the references in these works, and also [15], [16]).

The problem of plasma oscillations with diffuse boundary condition was considered in the works [17], [18] by method of integral transformations. In the works [19], [20] general asymptotic analysis of electric field behaviour at the large distance from the surface was carried out. In the work [19] particular significance of plasma behaviour analysis close to plasma resonance was shown. And in the same work [19] it was stated that plasma behaviour in this case for conditions of specular and diffuse electron scattering on the surface differs substantially.

In the works [22] general questions of this problem solvability were considered, but diffuse boundary conditions were taken into account. In the work [22] structure of discrete spectrum in dependence of parameters of the problem was analyzed. The detailed analysis of the solution in general case in the works mentioned above hasn't been carried out considering the complex character of this solution.

The present work is a continuation of electron plasma behaviour in external longitudinal alternating electric field research [22] – [26].

In the present paper the linearized problem of plasma oscillations in slab in external alternating electric field is solved analytically. Specular boundary conditions for electron reflection from the boundary are considered. In [24]–[26] diffuse boundary conditions were considered.

The coefficients of continuous and discrete spectra of the problem are obtained in the present work, which allows us to derive expressions for electron distribution function at the boundary of conductive medium and electric field in explicit form.

The present work is a continuation of our work [27], in which questions of half-space plasma waves reflection from the plane boundary bounding degenerate plasma were considered.

Let us note, that questions of plasma oscillations are also considered in nonlinear statement (see, for instance, the work [28], [29]).

## 2. PROBLEM STATEMENT

Let degenerate plasma occupy a slab (particularly, thin films)  $-a < x < a$ .

We take system of equations describing plasma behaviour. As a kinetic equation we take Boltzmann — Vlasov  $\tau$ -model kinetic equation:

$$\frac{\partial f}{\partial t} + \mathbf{v} \frac{\partial f}{\partial \mathbf{r}} + e \mathbf{E} \frac{\partial f}{\partial \mathbf{p}} = \frac{f_{eq}(\mathbf{r}, t) - f(\mathbf{r}, \mathbf{v}, t)}{\tau}. \quad (1.1)$$

Here  $f = f(\mathbf{r}, \mathbf{v}, t)$  is the electron distribution function,  $e$  is the electron charge,  $\mathbf{p} = m\mathbf{v}$  is the momentum of an electron,  $m$  is the electron mass,  $\tau$  is the character time between two collisions,  $\mathbf{E} = \mathbf{E}(\mathbf{r}, t)$  is the self-consistent electric field inside plasma,  $f_{eq} = f_{eq}(\mathbf{r}, t)$  is the local equilibrium

Fermi — Dirac distribution function,  $f_{eq} = \Theta(\mathcal{E}_F(t, x) - \mathcal{E})$ , where  $\Theta(x)$  is the function of Heaviside,

$$\Theta(x) = \begin{cases} 1, & x > 0, \\ 0, & x < 0, \end{cases}$$

$\mathcal{E}_F(t, x) = \frac{1}{2}mv_F^2(t, x)$  is the disturbed kinetic energy of Fermi,  $\mathcal{E} = \frac{1}{2}mv^2$  is the kinetic energy of electron.

Let us take the Maxwell equation for electric field

$$\text{div } \mathbf{E}(\mathbf{r}, t) = 4\pi\rho(\mathbf{r}, t). \quad (1.2)$$

Here  $\rho(\mathbf{r}, t)$  is the charge density,

$$\rho(\mathbf{r}, t) = e \int (f(\mathbf{r}, \mathbf{v}, t) - f_0(\mathbf{v})) d\Omega_F, \quad (1.3)$$

where

$$d\Omega_F = \frac{2d^3p}{(2\pi\hbar)^3}, \quad d^3p = dp_x dp_y dp_z.$$

Here  $f_0$  is the undisturbed Fermi — Dirac electron distribution function,

$$f_0(\mathcal{E}) = \Theta(\mathcal{E}_F - \mathcal{E}),$$

$\hbar$  is the Planck's constant,  $\nu$  is the effective frequency of electron collisions,  $\nu = 1/\tau$ ,  $\mathcal{E}_F = \frac{1}{2}mv_F^2$  is the undisturbed kinetic energy of Fermi,  $v_F$  is the electron velocity at the Fermi surface, which is supposed to be spherical.

We assume that external electric field outside the plasma is perpendicular to the plasma boundary and changes according to the following law:  $E_0 \exp(-i\omega t)$ .

Then one can consider that self-consistent electric field  $\mathbf{E}(\mathbf{r}, t)$  inside plasma has one  $x$ -component and changes only lengthwise the axis  $x$ :

$$\mathbf{E} = \{E_x(x, t), 0, 0\}.$$

Under this configuration the electric field is perpendicular to the boundary of plasma, which is situated in the plane  $x = 0$ .

We will linearize the local equilibrium Fermi — Dirac distribution  $f_{eq}$  in regard to the undisturbed distribution  $f_0(\mathcal{E})$ :

$$f_{eq} = f_0(\mathcal{E}) + [\mathcal{E}_F(x, t) - \mathcal{E}]\delta(\mathcal{E}_F - \mathcal{E}),$$

where  $\delta(x)$  is the delta – function of Dirac.

We also linearize the electron distribution function  $f$  in terms of absolute Fermi — Dirac distribution  $f_0(\mathcal{E})$ :

$$f = f_0(\mathcal{E}) + f_1(x, \mathbf{v}, t). \quad (1.4)$$

After the linearization of the equations (1.1)–(1.3) with the help of (1.4) we obtain the following system of equations:

$$\frac{\partial f_1}{\partial t} + v_x \frac{\partial f_1}{\partial x} + \nu f_1(x, \mathbf{v}, t) = \delta(\mathcal{E}_F - \mathcal{E}) [e E_x(x, t) v_x + \nu [\mathcal{E}_F(x, t) - \mathcal{E}_F]], \quad (1.5)$$

$$\frac{\partial E_x(x, t)}{\partial x} = \frac{8\pi e}{(2\pi\hbar)^3} \int f_1(x, \mathbf{v}', t) d^3 p' \quad (1.6)$$

From the law of preservation of number of particles

$$\int f_{eq} d\Omega_F = \int f d\Omega_F$$

we find:

$$[\mathcal{E}_F(x, t) - \mathcal{E}_F] \int \delta(\mathcal{E}_F - \mathcal{E}) d^3 p = \int f_1 d^3 p. \quad (1.7)$$

From the equation (1.5) it is seen that we should search for the function  $f_1$  in the form proportional to the delta – function:

$$f_1 = \mathcal{E}_F \delta(\mathcal{E}_F - \mathcal{E}) H(x, \mu, t), \quad \mu = \frac{v_x}{v}. \quad (1.8)$$

The system of equations (1.5) and (1.6) with the help of (1.7) and (1.8) can be transformed to the following form:

$$\begin{aligned} \frac{\partial H}{\partial t} + v_F \mu \frac{\partial H}{\partial x} + \nu H(x, \mu, t) &= \\ &= \frac{e v_F \mu}{\mathcal{E}_F} E_x(x, t) + \frac{\nu}{2} \int_{-1}^1 H(x, \mu', t) d\mu', \\ \frac{\partial E_x(x, t)}{\partial x} &= \frac{16\pi^2 e \mathcal{E}_F m^2 v_F}{(2\pi\hbar)^3} \int_{-1}^1 H(x, \mu', t) d\mu'. \end{aligned}$$

Further we introduce dimensionless function

$$e(x, t) = \frac{e v_F}{\nu \mathcal{E}_F} E_x(x, t)$$

and pass to dimensionless coordinate  $x_1 = x/l$ , where  $l = v_F\tau$  is the mean free path of electrons, and we introduce dimensionless time  $t_1 = \nu t$ . We obtain the following system of equations:

$$\frac{\partial H}{\partial t_1} + \mu \frac{\partial H}{\partial x_1} + \nu H(x_1, \mu, t_1) = \mu e(x_1, t_1) + \frac{1}{2} \int_{-1}^1 H(x_1, \mu', t_1) d\mu', \quad (1.9)$$

$$\frac{\partial e(x_1, t_1)}{\partial x_1} = \frac{3\omega_p^2}{2\nu^2} \int_{-1}^1 H(x_1, \mu', t_1) d\mu'. \quad (1.10)$$

Here  $\omega_p$  is the electron (Langmuir) frequency of plasma oscillations,

$$\omega_p^2 = \frac{4\pi e^2 N}{m},$$

$N$  is the numerical density (concentration),  $m$  is the electron mass.

We used the following well-known relation for degenerate plasma for the derivation of the equations (1.9) and (1.10)

$$\left(\frac{v_F m}{\hbar}\right)^3 = 3\pi^2 N.$$

Let  $k$  to be a dimensional wave number, and let us introduce dimensionless wave number  $k_1 = k \frac{v_F}{\omega_p}$ , then  $kx = \frac{k_1 x_1}{\varepsilon}$ , where  $\varepsilon = \frac{\nu}{\omega_p}$ . We introduce the quantity  $\omega_1 = \omega\tau = \frac{\omega}{\nu}$ .

### 3. BOUNDARY CONDITIONS STATEMENT

Let us outline the time variable of the functions  $H(x_1, \mu, t_1)$  and  $e(x_1, t_1)$ , assuming

$$H(x_1, \mu, t_1) = e^{-i\omega_1 t_1} h(x_1, \mu), \quad (2.1)$$

$$e(x_1, t_1) = e^{-i\omega_1 t_1} e(x_1). \quad (2.2)$$

The system of equations (1.9) and (1.10) in this case will be transformed to the following form:

$$\mu \frac{\partial h}{\partial x_1} + (1 - i\omega_1) h(x_1, \mu) = \mu e(x_1) + \frac{1}{2} \int_{-1}^1 h(x_1, \mu') d\mu', \quad (2.3)$$

$$\frac{de(x_1)}{dx_1} = \frac{3\omega_p^2}{2\nu^2} \int_{-1}^1 h(x_1, \mu') d\mu'. \quad (2.4)$$

Further instead of  $x_1, t_1$  we write  $x, t$ . We rewrite the system of equations (2.3) and (2.4) in the form:

$$\mu \frac{\partial h}{\partial x} + z_0 h(x, \mu) = \mu e(x) + \frac{1}{2} \int_{-1}^1 h(x, \mu') d\mu', \quad (2.5)$$

$$\frac{de(x)}{dx} = \frac{3}{2\varepsilon^2} \int_{-1}^1 h(x, \mu') d\mu'. \quad (2.6)$$

Here

$$z_0 = 1 - i\omega_1 = 1 - \frac{\omega}{\nu} = 1 - i\omega\tau = 1 - i\Omega, \quad \Omega = \omega\tau.$$

We consider the external electric field outside the plasma is perpendicular to the plasma boundary and changes according to the following law:  $E_0 \exp(-i\omega t)$ . This means that for the field inside plasma on the plasma boundary the following condition is satisfied:

$$e(-a_1) = e_s, \quad e(+a_1) = e_s, \quad (2.7)$$

where  $a_1 = a/l$  is the dimensionless width (depth) of slab. Further we will write  $a$  instead  $a_1$ .

We consider specular boundary conditions. For the function  $h(x, \mu)$  this conditions will be written in the following form:

$$h(-a, \mu) = h(-a, -\mu), \quad 0 < \mu < 1, \quad (2.8)$$

and

$$h(a, \mu) = h(a, -\mu), \quad -1 < \mu < 0. \quad (2.9)$$

Condition of symmetry of boundary conditions (2.7) – (2.9) and the equation (2.6) mean, that electric field  $e(x)$  and function  $h(x, \mu)$  in the slab possess properties of symmetry

$$e(x) = e(-x), \quad h(x, \mu) = h(x, -\mu). \quad (2.10)$$

The non-flowing condition for the particle (electric current) flow through the plasma boundary means that

$$\int_{-1}^1 \mu h(-a, \mu) d\mu = \int_{-1}^1 \mu h(a, \mu) d\mu = 0.$$

In accordance to (2.10) this condition it is carried out automatically.

The problem statement is completed. Now the problem consists in finding of such solution of the system of equations (2.5) and (2.6), which satisfies the boundary conditions (2.7) and (2.8). Further, with the use of the solution of the problem, it is required to built the profiles of the distribution function of the electrons moving to the plasma surface, and profile of the electric field.

## 5. SEPARATION OF VARIABLES AND CHARACTERISTIC SYSTEM

Application of the general Fourier method of the separation of variables in several steps results in the following substitution [31]:

$$h_\eta(x, \mu) = \exp\left(-\frac{z_0 x}{\eta}\right) \Phi_1(\eta, \mu) + \exp\left(\frac{z_0 x}{\eta}\right) \Phi_2(\eta, \mu), \quad (3.1)$$

$$e_\eta(x) = \left[ \exp\left(-\frac{z_0 x}{\eta}\right) + \exp\left(\frac{z_0 x}{\eta}\right) \right] E(\eta), \quad (3.2)$$

where  $\eta$  is the spectrum parameter or the parameter of separation, which is complex in general.

We substitute the equalities (3.1) and (3.2) into the equations (2.5) and (2.6). We obtain the following characteristic system of equations:

$$z_0(\eta - \mu) \Phi_1(\eta, \mu) = \eta \mu E(\eta) + \frac{\eta}{2} \int_{-1}^1 \Phi_1(\eta, \mu') d\mu', \quad (3.3)$$

$$z_0(\eta + \mu) \Phi_1(\eta, \mu) = \eta \mu E(\eta) + \frac{\eta}{2} \int_{-1}^1 \Phi_2(\eta, \mu') d\mu', \quad (3.4)$$

$$-\frac{z_0}{\eta} E(\eta) = \frac{3}{\varepsilon^2} \cdot \frac{1}{2} \int_{-1}^1 \Phi_1(\eta, \mu') d\mu', \quad (3.5)$$



$$\frac{z_0}{\eta}E(\eta) = \frac{3}{\varepsilon^2} \cdot \frac{\eta}{2} \int_{-1}^1 \Phi_2(\eta, \mu') d\mu'. \quad (3.6)$$

Let us introduce the designations:

$$\eta_1^2 = \frac{\varepsilon^2 z_0}{3}, \quad \varepsilon = \frac{\nu}{\omega_p}.$$

From equations (3.5) and (3.6) we obtain

$$\int_{-1}^1 \left[ \Phi_1(\eta, \mu) d\mu + \Phi_2(\eta, \mu) \right] d\mu = 0. \quad (3.7)$$

Let us introduce the designations

$$n(\eta) = \int_{-1}^1 \Phi_1(\eta, \mu) d\mu. \quad (3.8)$$

From equation (3.5) we obtain that

$$E(\eta) = -\frac{3}{2\varepsilon^2 z_0} \eta n(\eta), \quad (3.9)$$

whence

$$n(\eta) = -\frac{2\varepsilon^2 z_0}{3} \cdot \frac{E(\eta)}{\eta},$$

or

$$n(\eta) = -2\eta_1^2 \frac{E(\eta)}{\eta}.$$

By means of equalities (3.7) - (3.9) we will rewritten the equations (3.3) and (3.4)

$$(\eta - \mu)\Phi_1(\eta, \mu) = \frac{E(\eta)}{z_0}(\mu\eta - \eta_1^2), \quad (3.10)$$

$$(\eta + \mu)\Phi_2(\eta, \mu) = \frac{E(\eta)}{z_0}(\mu\eta + \eta_1^2). \quad (3.11)$$

Solution of the system (3.10) and (3.11) depends essentially on the condition whether the spectrum parameter  $\eta$  belongs to the interval  $-1 < \eta < 1$ . In connection with this the interval  $-1 < \eta < 1$  we will call as continuous spectrum of the characteristic system.

Let the parameter  $\eta \in (-1, 1)$ . Then from the equations (3.10) and (3.11) in the class of general functions we will find eigenfunction corresponding to the continuous spectrum

$$\Phi_1(\eta, \mu) = \frac{E(\eta)}{z_0} P \frac{\mu\eta - \eta_1^2}{\eta - \mu} + g_1(\eta) \delta(\eta - \mu), \quad (3.12)$$

$$\Phi_2(\eta, \mu) = \frac{E(\eta)}{z_0} P \frac{\mu\eta + \eta_1^2}{\eta + \mu} + g_2(\eta) \delta(\eta - \mu). \quad (3.13)$$

In these equations (3.12) and (3.13)  $\delta(x)$  is the delta-function of Dirac, the symbol  $Px^{-1}$  means the principal value of the integral under integrating of the expression  $x^{-1}$ .

Substituting now (3.12) and (3.13) in the equations (3.5) and (3.6), We receive the equations

$$\begin{aligned} \frac{E(\eta)}{z_0} \left[ 1 - \frac{\eta}{2c} \int_{-1}^1 \frac{\mu'\eta - \eta_1^2}{\mu' - \eta} d\mu' \right] &= -\frac{\eta}{2c} g_1(\eta), \\ \frac{E(\eta)}{z_0} \left[ 1 + \frac{\eta}{2c} \int_{-1}^1 \frac{\mu'\eta + \eta_1^2}{\mu' + \eta} d\mu' \right] &= \frac{\eta}{2c} g_1(\eta), \end{aligned}$$

from which we obtain

$$g_1(\eta) = -2\eta_1^2 \frac{\lambda(\eta)}{\eta} E(\eta), \quad g_2(\eta) = -g_1(\eta) = 2\eta_1^2 \frac{\lambda(\eta)}{\eta} E(\eta). \quad (3.14)$$

Here dispersive function is entered

$$\lambda(z) = 1 - \frac{z}{2c} \int_{-1}^1 \frac{\mu z - \eta_1^2}{\mu - z} d\mu, \quad (3.15)$$

where

$$c = \frac{\varepsilon^2 z_0^2}{3} = z_0 \eta_1^2.$$

Functions (3.12) and (3.13) are called eigen functions of the continuous spectrum, since the spectrum parameter  $\eta$  fills out the continuum  $(-1, +1)$  compactly. The eigen solutions of the given problem can be found from the equalities (3.1) and (3.2).

The dispersion function  $\lambda(z)$  we express in the terms of the Case dispersion function [31]:

$$\lambda(z) = 1 - \frac{1}{z_0} + \frac{1}{z_0} \left(1 - \frac{z^2}{\eta_1^2}\right) \lambda_c(z),$$

where

$$\lambda_c(z) = 1 + \frac{z}{2} \int_{-1}^1 \frac{d\tau}{\tau - z} = \frac{1}{2} \int_{-1}^1 \frac{\tau d\tau}{\tau - z}$$

is the Case dispersion function [31].

The boundary values of the dispersion function from above and below the cut (interval  $(-1, 1)$ ) we define in the following way:

$$\lambda^\pm(\mu) = \lim_{\varepsilon \rightarrow 0, \varepsilon > 0} \lambda(\mu \pm i\varepsilon), \quad \mu \in (-1, 1).$$

The boundary values of the dispersion function from above and below the cut are calculated according to the Sokhotzky formulas

$$\lambda^\pm(\mu) = \lambda(\mu) \pm \frac{i\pi\mu}{2\eta_1^2 z_0} (\eta_1^2 - \mu^2), \quad -1 < \mu < 1,$$

from where

$$\begin{aligned} \lambda^+(\mu) - \lambda^-(\mu) &= \frac{i\pi}{\eta_1^2 z_0} \mu (\eta_1^2 - \mu^2), \\ \frac{\lambda^+(\mu) + \lambda^-(\mu)}{2} &= \lambda(\mu), \quad -1 < \mu < 1, \end{aligned}$$

where

$$\lambda(\mu) = 1 + \frac{\mu}{2\eta_1^2 z_0} \int_{-1}^1 \frac{\eta_1^2 - \eta^2}{\eta - \mu} d\eta,$$

and the integral in this equality is understood as singular in terms of the principal value by Cauchy. Besides that, the function  $\lambda(\mu)$  can be represented in the following form:

$$\begin{aligned} \lambda(\mu) &= 1 - \frac{1}{z_0} + \frac{1}{z_0} \left(1 - \frac{\mu^2}{\eta_1^2}\right) \lambda_c(\mu), \\ \lambda_c(\mu) &= 1 + \frac{\mu}{2} \ln \frac{1 - \mu}{1 + \mu}. \end{aligned}$$

Substituting relations (3.14) in (3.12) and (3.13), we will present last expressions in the following form

$$\Phi_1(\eta, \mu) = \frac{E(\eta)}{z_0} \left[ P \frac{\mu\eta - \eta_1^2}{\eta - \mu} - 2c \frac{\lambda(\eta)}{\eta} \delta(\eta - \mu) \right],$$

and

$$\Phi_2(\eta, \mu) = \frac{E(\eta)}{z_0} \left[ P \frac{\mu\eta + \eta_1^2}{\eta + \mu} + 2c \frac{\lambda(\eta)}{\eta} \delta(\eta + \mu) \right],$$

or

$$\Phi_1(\eta, \mu) = \frac{E(\eta)}{z_0} F(\eta, \mu), \quad \Phi_2(\eta, \mu) = \frac{E(\eta)}{z_0} F(-\eta, \mu),$$

where

$$F(\eta, \mu) = P \frac{\mu\eta - \eta_1^2}{\eta - \mu} - 2c \frac{\lambda(\eta)}{\eta} \delta(\eta - \mu),$$

and

$$F(-\eta, \mu) = P \frac{\mu\eta + \eta_1^2}{\eta + \mu} + 2c \frac{\lambda(\eta)}{\eta} \delta(\eta + \mu).$$

It will be necessary for us the following relation of symmetry

$$F(-\eta, -\mu) = -F(\eta, \mu).$$

Let us notice, that eigen function  $F(\eta, \mu)$  satisfies to following condition of normalization

$$\int_{-1}^1 F(\eta, \mu) d\mu = -2cP \frac{1}{\eta},$$

and that analogous

$$\int_{-1}^1 F(-\eta, \mu) d\mu = 2cP \frac{1}{\eta}.$$

So, eigen function of a continuous spectrum is constructed and it is defined by equality

$$h_\eta(x, \mu) = \left[ \exp\left(-\frac{xz_0}{\eta}\right) F(\eta, \mu) + \exp\left(\frac{xz_0}{\eta}\right) F(-\eta, \mu) \right] \frac{E(\eta)}{z_0},$$

or, in explicit form,

$$h_\eta(x, \mu) = \left\{ \left[ \exp\left(-\frac{xz_0}{\eta}\right) \frac{\mu\eta - \eta_1^2}{\eta - \mu} + \exp\left(\frac{xz_0}{\eta}\right) \frac{\mu\eta + \eta_1^2}{\eta + \mu} \right] - \right.$$

$$-2c \frac{\lambda(\eta)}{\eta} \left[ \exp \left( -\frac{xz_0}{\eta} \right) \delta(\eta - \mu) - \exp \left( \frac{xz_0}{\eta} \right) \delta(\eta + \mu) \right] \left\} \frac{E(\eta)}{z_0}.$$

Let us replace exponents by hyperbolic functions and also we will transform both square brackets from the previous expression. As a result we receive, that

$$\begin{aligned} h_\eta(x, \mu) = \frac{2E(\eta)}{z_0} & \left\{ \operatorname{ch} \frac{xz_0}{\eta} \left[ P \left( \frac{\mu\eta - \eta_1^2}{\eta - \mu} + \frac{\mu\eta + \eta_1^2}{\eta + \mu} \right) + \right. \right. \\ & + 2c \frac{\lambda(\eta)}{\eta} (-\delta(\eta - \mu) + \delta(\eta + \mu)) \left. \right] + \operatorname{sh} \frac{xz_0}{\eta} \left[ P \left( -\frac{\mu\eta - \eta_1^2}{\eta - \mu} + \frac{\mu\eta + \eta_1^2}{\eta + \mu} \right) + \right. \\ & \left. \left. + 2c \frac{\lambda(\eta)}{\eta} (\delta(\eta - \mu) + \delta(\eta + \mu)) \right] \right\}. \end{aligned}$$

We will designate further

$$\begin{aligned} \varphi(\eta, \mu) = F(\eta, \mu) + F(-\eta, \mu) &= P \left( \frac{\mu\eta - \eta_1^2}{\eta - \mu} + \frac{\mu\eta + \eta_1^2}{\eta + \mu} \right) = \\ &= 2P \frac{\mu(\eta^2 - \eta_1^2)}{\eta^2 - \mu^2}, \end{aligned}$$

and

$$\begin{aligned} \psi(\eta, \mu) = -F(\eta, \mu) + F(-\eta, \mu) &= P \left( -\frac{\mu\eta - \eta_1^2}{\eta - \mu} + \frac{\mu\eta + \eta_1^2}{\eta + \mu} \right) = \\ &= 2P \frac{\eta(\mu^2 - \eta_1^2)}{\eta^2 - \mu^2}. \end{aligned}$$

Thus, eigen function of a continuous spectrum it is possible to present in the form of a linear combination of a hyperbolic sine and cosine

$$\begin{aligned} h_\eta(x, \mu) = \frac{2E(\eta)}{z_0} & \left\{ \operatorname{ch} \frac{xz_0}{\eta} \left[ P \frac{\mu(\eta^2 - \eta_1^2)}{\eta^2 - \mu^2} - c \frac{\lambda(\eta)}{\eta} (\delta(\eta - \mu) - \delta(\eta + \mu)) \right] - \right. \\ & \left. - \operatorname{sh} \frac{xz_0}{\eta} \left[ P \frac{\eta(\mu^2 - \eta_1^2)}{\eta^2 - \mu^2} - c \frac{\lambda(\eta)}{\eta} (\delta(\eta - \mu) + \delta(\eta + \mu)) \right] \right\}. \end{aligned}$$

Let us notice, that eigen functions of a continuous spectrum it is possible to present and in such form

$$h_\eta(x, \mu) = \frac{E(\eta)}{z_0} \left[ \operatorname{ch} \frac{xz_0}{\eta} \left( F(\eta, \mu) + F(-\eta, \mu) \right) + \right.$$

$$+ \operatorname{sh} \frac{xz_0}{\eta} \left( -F(\eta, \mu) + F(-\eta, \mu) \right) \Bigg]. \quad (3.16)$$

## 6. EIGENFUNCTIONS OF THE DISCRETE SPECTRUM

According to the definition, the discrete spectrum of the characteristic equation is a set of zeroes of the dispersion equation

$$\frac{\lambda(z)}{z} = 0. \quad (4.1)$$

We start to search zeroes of the equation (4.1). Let us take Laurent series of the dispersion function:

$$\lambda(z) = \lambda_\infty + \frac{\lambda_2}{z^2} + \frac{\lambda_4}{z^4} + \dots, \quad |z| > 1. \quad (4.2)$$

Here

$$\begin{aligned} \lambda_\infty &\equiv \lambda(\infty) = 1 - \frac{1}{z_0} + \frac{1}{3z_0\eta_1^2}, \\ \lambda_2 &= -\frac{1}{z_0} \left( \frac{1}{3} - \frac{1}{5\eta_1^2} \right), \\ \lambda_4 &= -\frac{1}{z_0} \left( \frac{1}{5} - \frac{1}{7\eta_1^2} \right). \end{aligned}$$

We express these parameters through the parameters  $\Omega$  and  $\varepsilon$ :

$$\begin{aligned} \lambda_\infty &\equiv \lambda(\infty) = \frac{2(\Omega - 1) + i\varepsilon + (\Omega - 1)(\Omega - 1 + i\varepsilon)}{(\Omega + i\varepsilon)^2}, \\ \lambda_2 &= -\frac{9 + 5i\varepsilon(\Omega + i\varepsilon)}{15(\Omega + i\varepsilon)^2}, \\ \lambda_4 &= -\frac{15 + 7i\varepsilon(\Omega + i\varepsilon)}{35(\Omega + i\varepsilon)^2}. \end{aligned}$$

It is easy seen that the dispersion function (3.15) in collisional plasma (i.e. when  $\varepsilon > 0$ ) in the infinity has the value which doesn't equal to zero:  $\lambda_\infty = \lambda(\infty) \neq 0$ .

Hence, the dispersion equation has infinity as a zero  $\eta_i = \infty$ , to which the discrete eigensolutions of the given system correspond:

$$h_\infty(x, \mu) = \frac{\mu}{z_0}, \quad e_\infty(x) = 1.$$

This solution is naturally called as mode of Drude. It describes the volume conductivity of metal, considered by Drude (see, for example, [32]).

Let us consider the question of the plasma mode existence in details. We find finite complex zeroes of the dispersion function. We use the principle of argument. We take the contour (see Fig. 1)

$$\Gamma_{\varepsilon}^{+} = \Gamma_R \cup \gamma_{\varepsilon},$$

which is passed in the positive direction and which bounds the biconnected domain  $D_R$ . This contour consists of the circumference

$$\Gamma_R = \{z : |z| = R, \quad R = \frac{1}{\varepsilon}, \quad \varepsilon > 0\},$$

and the contour  $\gamma_{\varepsilon}$ , which includes the cut  $[-1, +1]$ , and stands at the distance of  $\varepsilon$  from it.

Let us note that the dispersion function has not any poles in the domain  $D_R$ . Then according to the principle of argument the number [33] of zeroes  $N$  in the domain  $D_R$  equals to:

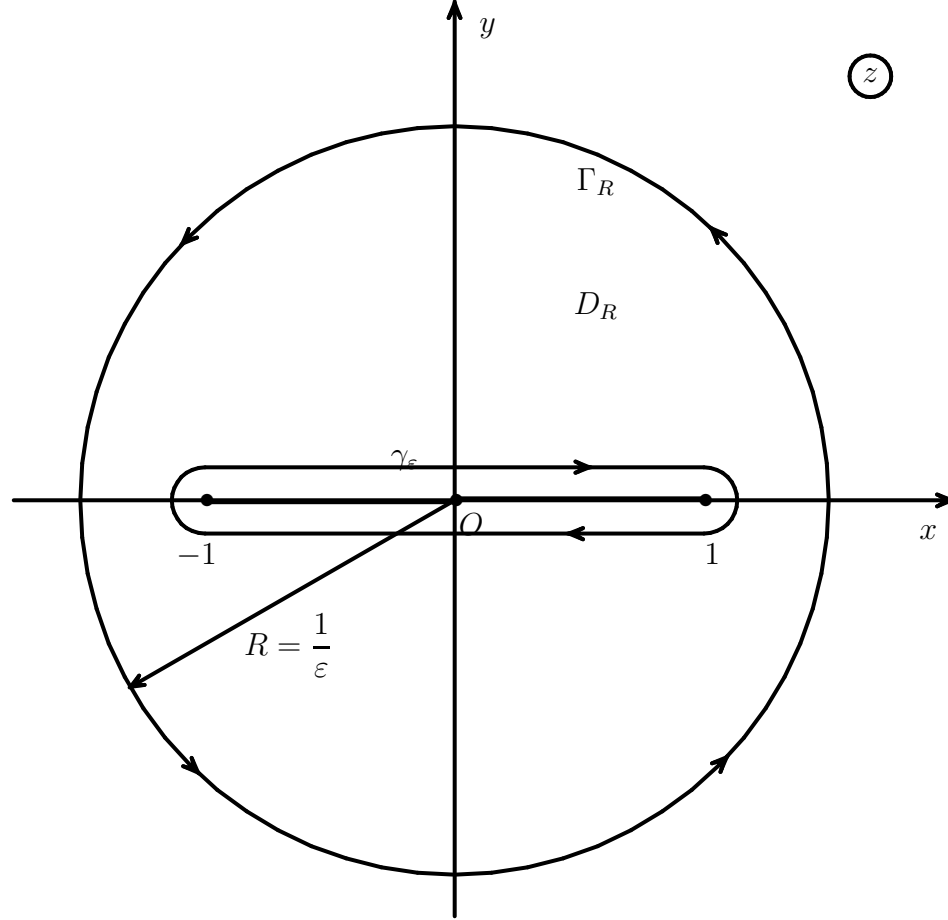
$$N = \frac{1}{2\pi i} \oint_{\Gamma_{\varepsilon}} d \ln \lambda(z).$$

Considering the limit in this equality when  $\varepsilon \rightarrow 0$  and taking into account that the dispersion function is analytic in the neighbourhood of the infinity, we obtain that

$$\begin{aligned} N &= \frac{1}{2\pi i} \int_{-1}^1 d \ln \lambda^{+}(\tau) - \frac{1}{2\pi i} \int_{-1}^1 d \ln \lambda^{-}(\tau) \\ &= \frac{1}{2\pi i} \int_{-1}^1 d \ln \frac{\lambda^{+}(\mu)}{\lambda^{-}(\mu)}. \end{aligned}$$

So, we obtained that

$$N = \frac{1}{2\pi i} \int_{-1}^1 d \ln \frac{\lambda^{+}(\tau)}{\lambda^{-}(\tau)}.$$



**Fig 1.** Contour  $\Gamma_\epsilon = \Gamma_R \cup \gamma_\epsilon$  for calculations of number of zero of dispersion function.



We divide this integral into two integrals by segments  $[-1, 0]$  and  $[0, 1]$ . In the first integral by the segment  $[-1, 0]$  we carry out replacement of variable  $\tau \rightarrow -\tau$ . Taking into account that  $\lambda^+(-\tau) = \lambda^-(\tau)$ , we obtain that

$$N = \frac{1}{2\pi i} \int_{-1}^1 d \ln \frac{\lambda^+(\tau)}{\lambda^-(\tau)} = \frac{1}{\pi i} \int_0^1 d \ln \frac{\lambda^+(\tau)}{\lambda^-(\tau)} = \frac{1}{\pi} \arg \frac{\lambda^+(\tau)}{\lambda^-(\tau)} \Big|_0^1. \quad (4.3)$$

Here under symbol  $\arg G(\tau) = \arg \frac{\lambda^+(\tau)}{\lambda^-(\tau)}$  we understand the regular branch of the argument, fixed in zero with the condition:  $\arg G(0) = 0$ .

We consider the curve

$$\gamma = \{z : z = G(\tau), \quad 0 \leq \tau \leq +1\},$$

where

$$G(\tau) = \frac{\lambda^+(\tau)}{\lambda^-(\tau)}.$$

It is obvious that

$$G(0) = 1, \quad \lim_{\tau \rightarrow +1} G(\tau) = 1.$$

Consequently, according to (4.3), the number of values  $N$  equals to doubled number of turns of the curve  $\gamma$  around the point of origin, i.e.

$$N = 2\kappa(G),$$

where

$$\kappa(G) = \text{Ind}_{[0, +1]} G(\tau)$$

is the index of the function  $G(\tau)$ .

Thus, the number of zeroes of the dispersion function, which are situated in complex plane outside of the segment  $[-1, 1]$  of the real axis, equals to doubled index of the function  $G(\tau)$ , calculated on the "semi-segment"  $[0, +1]$ .

Let us single real and imaginary parts of the function  $G(\mu)$  out. At first, we represent the function  $G(\mu)$  in the form:

$$G(\mu) = \frac{(z_0 - 1)\eta_1^2 + (\eta_1^2 - \mu^2)\lambda_0(\mu) + is(\mu)(\eta_1^2 - \mu^2)}{(z_0 - 1)\eta_1^2 + (\eta_1^2 - \mu^2)\lambda_0(\mu) - is(\mu)(\eta_1^2 - \mu^2)}.$$

where

$$s(\mu) = \frac{\pi}{2}\mu,$$

$$\lambda(\mu) = 1 - \frac{1}{z_0} + \frac{1}{z_0} \left(1 - \frac{\mu^2}{\eta_1^2}\right) \lambda_c(\mu),$$

and

$$\lambda_c(\mu) = 1 + \frac{\mu}{2} \ln \frac{1-\mu}{1+\mu}$$

is the dispersion function of Case, calculated on the cut (i.e., in the interval  $(-1, 1)$ ).

Taking into account that

$$z_0 - 1 = -i\frac{\omega}{\nu} = -i\frac{\Omega}{\varepsilon}, \quad \eta_1^2 = \frac{\varepsilon z_0}{3} = \frac{\varepsilon^2}{3} - i\frac{\varepsilon\Omega}{3},$$

$$(z_0 - 1)\eta_1^2 = -\frac{\Omega^2}{3} - i\frac{\varepsilon\Omega}{3},$$

we obtain

$$G(\mu) = \frac{P^-(\mu) + iQ^-(\mu)}{P^+(\mu) + iQ^+(\mu)},$$

where

$$P^\pm(\mu) = \Omega^2 - \lambda_0(\mu)(\varepsilon^2 - 3\mu^2) \pm \varepsilon\Omega s(\mu),$$

$$Q^\pm(\mu) = \varepsilon\Omega(1 + \lambda_0(\mu)) \pm s(\mu)(\varepsilon^2 - 3\mu^2).$$

Now we can easily single real and imaginary parts of the function  $G(\mu)$  out:

$$G(\mu) = \frac{g_1(\mu)}{g(\mu)} + i\frac{g_2(\mu)}{g(\mu)}.$$

Here

$$\begin{aligned} g(\mu) &= [P^+(\mu)]^2 + [Q^+(\mu)]^2 = [\Omega^2 + \lambda_0(3\mu^2 - \varepsilon^2) - \\ &\quad - \varepsilon\Omega s]^2 + [\varepsilon\Omega(1 + \lambda_0) - s(3\mu^2 - \varepsilon^2)]^2, \\ g_1(\mu) &= P^+(\mu)P^-(\mu) + Q^+(\mu)Q^-(\mu) = [\Omega^2 + \lambda_0(3\mu^2 - \varepsilon^2)]^2 - \\ &\quad - \varepsilon^2\Omega^2[s^2 - (1 + \lambda_0)^2] - (3\mu^2 - \varepsilon^2)^2 s^2, \\ g_2(\mu) &= P^+(\mu)Q^-(\mu) - P^-(\mu)Q^+(\mu) = 2s[\Omega^2(3\mu^2 - \varepsilon^2) + \\ &\quad + \lambda_0(3\mu^2 - \varepsilon^2)^2 + \varepsilon^2\Omega^2(1 + \lambda_0)], \end{aligned}$$

We consider (see Fig. 2) the curve  $L$ , which is defined in implicit form by the following parametric equations:

$$L = \{(\Omega, \varepsilon) : g_1(\mu; \Omega, \varepsilon) = 0, \quad g_2(\mu; \Omega, \varepsilon) = 0, \quad 0 \leq \mu \leq 1\},$$

and which lays in the plane of the parameters of the problem  $(\Omega, \varepsilon)$ , and when passing through this curve the index of the function  $G(\mu)$  at the positive "semi-segment"  $[0, 1]$  changes stepwise.

From the equation  $g_2 = 0$  we find:

$$\Omega^2 = -\frac{\lambda_0(\mu)(3\mu^2 - \varepsilon^2)}{3\mu^2 + \varepsilon^2\lambda_0(\mu)}. \quad (4.4)$$

Now from the equation  $g_1 = 0$  with the help of (4.4) we find that

$$\varepsilon = \sqrt{L_2(\mu)}, \quad (4.5)$$

where

$$L_2(\mu) = -\frac{3\mu^2 s^2(\mu)}{\lambda_0(\mu)[s^2(\mu) + (1 + \lambda_0(\mu))^2]}.$$

Substituting (4.5) into (4.4), we obtain:

$$\Omega = +\sqrt{L_1(\mu)}, \quad (4.6)$$

where

$$L_1(\mu) = -\frac{3\mu^2[s^2(\mu) + \lambda_0(\mu)(1 + \lambda_0(\mu))]^2}{\lambda_0(\mu)[s^2(\mu) + (1 + \lambda_0(\mu))^2]}.$$

Functions (4.5) and (4.6) determine the curve  $L$  which is the border of the domain  $D^+$  (we designate the external area to the domain as  $D^-$ ) in explicit parametrical form (see Fig. 2). As in the work [34] we can prove that if  $(\gamma, \varepsilon) \in D^+$ , then  $\varkappa(G) = \text{Ind}_{[0, +1]} G(\mu) = 1$  (the curve  $L$  encircles the point of origin once), and if  $(\gamma, \varepsilon) \in D^-$ , then  $\varkappa(G) = \text{Ind}_{[0, +1]} G(\mu) = 0$  (the curve  $L$  doesn't encircle the point of origin).

We note, that in the work [34] the method of analysis of boundary regime when  $(\Omega, \varepsilon) \in L$  was developed.

From the expression (3.2) one can see that the number of zeroes of the dispersion function equals to two if  $(\Omega, \varepsilon) \in D^+$ , and equals to zero if  $(\Omega, \varepsilon) \in D^-$ .

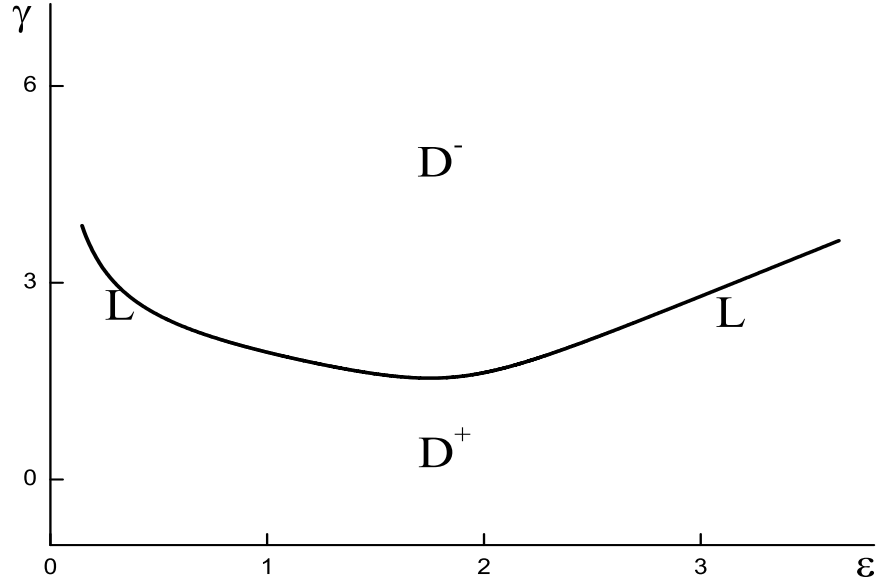


Fig. 2.

Since the dispersion function is even its zeroes differ from each other by sign. We designate these zeroes as following  $\pm\eta_0$ , by  $\eta_0$  we take the zero which satisfies the condition  $\text{Re } \eta_0 > 0$ . The following solution corresponds to the zero  $\eta_0$

$$h_{\eta_0}(x, \mu) = \text{ch } \frac{xz_0}{\eta_0} [F(\eta_0, \mu) + F(-\eta_0, \mu)] + \\ + \text{sh } \frac{xz_0}{\eta_0} [-F(\eta_0, \mu) + F(-\eta_0, \mu)], \quad (4.7)$$

$$e_{\eta_0}(x) = 2 \text{ch } \frac{z_0 x}{\eta_0}. \quad (4.8)$$

Here

$$F(\eta_0, \mu) = \frac{\eta_0 \mu - \eta_1^2}{\eta_0 - \mu}, \quad F(-\eta_0, \mu) = \frac{\eta_0 \mu + \eta_1^2}{\eta_0 + \mu}.$$

It is easy to see, that function  $h_{\eta_0}(x, \mu)$  is even on  $\eta_0$

$$h_{\eta_0}(x, \mu) = h_{-\eta_0}(x, \mu).$$

Function  $h_{\eta_0}(x, \mu)$  we will present in the explicit form

$$h_{\eta_0}(x, \mu) = \text{ch} \frac{xz_0}{\eta_0} \left[ \frac{\eta_0\mu - \eta_1^2}{\eta_0 - \mu} + \frac{\eta_0\mu + \eta_1^2}{\eta_0 + \mu} \right] + \\ + \text{sh} \frac{xz_0}{\eta_0} \left[ -\frac{\eta_0\mu - \eta_1^2}{\eta_0 - \mu} + \frac{\eta_0\mu + \eta_1^2}{\eta_0 + \mu} \right], \quad (4.9)$$

or

$$h_{\eta_0}(x, \mu) = \text{ch} \frac{xz_0}{\eta_0} \varphi(\eta_0, \mu) + \text{sh} \frac{xz_0}{\eta_0} \psi(\eta_0, \mu). \quad (4.9')$$

Here

$$\begin{aligned} \varphi(\eta_0, \mu) &= F(\eta_0, \mu) + F(-\eta_0, \mu) = \\ &= \frac{\eta_0\mu - \eta_1^2}{\eta_0 - \mu} + \frac{\eta_0\mu + \eta_1^2}{\eta_0 + \mu} = \frac{2\mu(\eta_0^2 - \eta_1^2)}{\eta_0^2 - \mu^2}, \\ \psi(\eta_0, \mu) &= -F(\eta_0, \mu) + F(-\eta_0, \mu) = \\ &= -\frac{\eta_0\mu - \eta_1^2}{\eta_0 - \mu} + \frac{\eta_0\mu + \eta_1^2}{\eta_0 + \mu} = \frac{2\eta_0(\eta_1^2 - \mu^2)}{\eta_0^2 - \mu^2}. \end{aligned}$$

This solution is naturally called as mode of Debay (this is plasma mode). In the case of low frequencies it describes well-known screening of Debay [3]. The external field penetrates into plasma on the depth of  $r_D$ ,  $r_D$  is the raduis of Debay. When the external field frequencies are close to Langmuir frequencies, the mode of Debay describes plasma oscillations (see, for instance, [3, 32]).

**Note 5.1.** If to enter expression  $c/z$  "inside" of expression of dispersion function we will receive expression for dispersion function  $h(z)$  from our article [21]

$$\begin{aligned} h(z) &= \frac{c}{z} \lambda(z) = \frac{c}{z} - \frac{1}{2} \int_{-1}^1 \frac{\mu z - \eta_1^2}{\mu - z} d\mu = \\ &= \frac{c}{z} - z - (z^2 - \eta_1^2) \frac{1}{2} \ln \frac{z-1}{z+1}. \end{aligned}$$

## 6. EXPANSIONS IN THE TERMS OF EIGEN FUNCTIONS

We will seek for the solution of the system of equations (2.5) and (2.6) with boundary conditions (2.7)–(2.9) in the form of linear combination of discrete eigen solutions of the characteristic system and integral taken over continuous spectrum of the system. Let us prove that the following theorem is true.

**Theorem 6.1.** *System of equations (2.5) and (2.6) with boundary conditions (3.1), (3.6) and (2.7) has a unique solution, which can be presented as an expansion by eigen functions of the characteristic system:*

$$\begin{aligned}
 h(x, \mu) = & \frac{E_\infty}{z_0} \mu + \\
 & + \frac{E_0}{z_0} \left[ \operatorname{ch} \frac{x z_0}{\eta_0} (F(\eta_0, \mu) + F(-\eta_0, \mu)) + \operatorname{sh} \frac{x z_0}{\eta_0} (-F(\eta_0, \mu) + F(-\eta_0, \mu)) \right] + \\
 & + \int_{-1}^1 \left\{ \operatorname{ch} \frac{z_0 x}{\eta} [F(\eta, \mu) + F(-\eta, \mu)] + \operatorname{sh} \frac{x z_0}{\eta} [-F(\eta, \mu) + F(-\eta, \mu)] \right\} \frac{E(\eta)}{z_0} d\eta,
 \end{aligned} \tag{6.1}$$

$$e(x) = E_\infty + 2E_0 \operatorname{ch} \frac{z_0 x}{\eta} + 2 \int_{-1}^1 \operatorname{ch} \frac{z_0 x}{\eta} E(\eta) d\eta. \tag{6.2}$$

Here  $E_0$  and  $E_\infty$  is unknown coefficients corresponding to the discrete spectrum ( $E_0$  is the amplitude of Debay,  $E_1$  is the amplitude of Drude),  $E(\eta)$  is unknown function, which is called as coefficient of discrete spectrum.

When  $(\Omega, \varepsilon) \in D^-$  in expansions (6.1) and (6.2) we should take  $E_0 = 0$ .

Further we will consider the following case  $(\Omega, \varepsilon) \in D^+$ .

Our purpose is to find the coefficient of the continuous spectrum, coefficients of the discrete spectrum and to built expressions for electron distribution function at the plasma surface and electric field.

**Proof.** Let us consider expansion (6.1), we will replace in it  $\mu$  on  $-\mu$ . Then we will substitute  $h(a, \mu)$  and  $h(a, -\mu)$  in The equation  $\frac{1}{2}[h(a, \mu) - h(a, -\mu)] = 0$ . After variety of transformations let us have

$$E_\infty \mu + E_0 \operatorname{ch} \frac{a z_0}{\eta_0} [F(\eta_0, \mu) + F(-\eta_0, \mu)] +$$

$$+ \int_{-1}^1 \operatorname{ch} \frac{az_0}{\eta} [F(\eta, \mu) + F(-\eta, \mu)] E(\eta) d\eta = 0, \quad -1 < \mu < 1, \quad (6.3)$$

or

$$E_\infty \mu + E_0 \operatorname{ch} \frac{az_0}{\eta_0} \varphi(\eta_0, \mu) + 2 \int_{-1}^1 \operatorname{ch} \frac{az_0}{\eta} F(\eta, \mu) E(\eta) d\eta = 0. \quad (6.4)$$

Let us pass from integral Fredholm equation (6.4) to singular integral equation with Cauchy kernel, having substituted in (6.4) obvious representation  $F(\eta, \mu)$

$$E_\infty \mu + E_0 \operatorname{ch} \frac{az_0}{\eta_0} \varphi(\eta_0, \mu) + 2 \int_{-1}^1 \frac{\mu\eta - \eta_1^2}{\eta - \mu} E(\eta) \operatorname{ch} \left( \frac{z_0 a}{\eta} \right) d\eta -$$

$$- 4c \frac{\lambda(\mu)}{\mu} E(\mu) \operatorname{ch} \frac{z_0 a}{\mu} = 0, \quad 0 < \mu < 1. \quad (6.5)$$

It is easy to check up, that function

$$M(z) = \int_{-1}^1 \frac{z\eta - \eta_1^2}{\eta - z} E(\eta) \operatorname{ch} \frac{z_0 a}{\eta} d\eta \quad (6.6)$$

is odd. Besides, all members of the equation (6.5) are odd on  $\mu$ .

Extending the function  $E(\eta)$  into the interval  $(-1, 0)$  evenly, so that  $E(\eta) = E(-\eta)$ , and extending the equation into the interval  $(-1; 1)$  unevenly, we transform the equation (6.5) to the form

$$E_\infty \mu + E_0 \operatorname{ch} \frac{az_0}{\eta_0} \varphi(\eta_0, \mu) + 2 \int_{-1}^1 \operatorname{ch} \left( \frac{z_0 a}{\eta} \right) \frac{\mu\eta - \eta_1^2}{\eta - \mu} E(\eta) d\eta -$$

$$- 4c \frac{\lambda(\mu)}{\mu} \operatorname{ch} \left( \frac{z_0 a}{\mu} \right) E(\mu) = 0, \quad -1 < \mu < 1. \quad (6.7)$$

Let us reduce the equation (6.7) to boundary value problem of Riemann — Hilbert. For this purpose we will take formulas of Sohotsky for the auxiliary functions  $M(z)$  and dispersion function  $\lambda(z)$

$$M^+(\mu) - M^-(\mu) = 2\pi i \operatorname{ch} \left( \frac{z_0 a}{\mu} \right) (\mu^2 - \eta_1^2) E(\mu), \quad -1 < \mu < 1, \quad (6.8)$$

$$\frac{M^+(\mu) + M^-(\mu)}{2} = M(\mu), \quad -1 < \mu < 1,$$

where

$$M(\mu) = \int_{-1}^1 \frac{\mu\eta - \eta_1^2}{\eta - \mu} \operatorname{ch} \left( \frac{z_0 a}{\eta} \right) E(\eta) d\eta,$$

and last integral is understood as singular in sense integral principal value on Cauchy, and

$$\lambda^+(\mu) - \lambda^-(\mu) = -\frac{i\pi}{c} \mu(\mu^2 - \eta_1^2), \quad -1 < \mu < 1,$$

$$\frac{\lambda^+(\mu) + \lambda^-(\mu)}{2} = \lambda(\mu), \quad -1 < \mu < 1,$$

where

$$\lambda(\mu) = 1 - \frac{\mu}{2c} \int_{-1}^1 \frac{\mu'\mu - \eta_1^2}{\mu' - \mu} d\mu'.$$

As a result of use of last formulas we will come to the boundary value problem

$$\begin{aligned} E_\infty \mu + E_0 \operatorname{ch} \frac{az_0}{\eta_0} \varphi(\eta_0, \mu) + [M^+(\mu) + M^-(\mu)] + \\ + \frac{\lambda^+(\mu) + \lambda^-(\mu)}{\lambda^+(\mu) - \lambda^-(\mu)} [M^+(\mu) - M^-(\mu)] = 0. \end{aligned}$$

Let us transform this equation to the form

$$\begin{aligned} (M^+ + M^-)(\lambda^+ - \lambda^-) + (M^+ - M^-)(\lambda^+ + \lambda^-) + \\ + (\lambda^+ - \lambda^-)(E_\infty \mu + E_0 \operatorname{ch} \frac{az_0}{\eta_0} \varphi(\eta_0, \mu)) = 0, \quad -1 < \mu < 1. \end{aligned}$$

From here we receive the boundary condition of boundary value problem of Riemann — Hilbert

$$\begin{aligned} \lambda^+(\mu) \left[ M^+(\mu) + \frac{E_\infty}{2} \mu + \frac{E_0}{2} \operatorname{ch} \frac{az_0}{\eta_0} \varphi(\eta_0, \mu) \right] - \\ = \lambda^-(\mu) \left[ M^-(\mu) + \frac{E_\infty}{2} \mu + \frac{E_0}{2} \operatorname{ch} \frac{az_0}{\eta_0} \varphi(\eta_0, \mu) \right] = 0, \quad -1 < \mu < 1. \quad (6.9) \end{aligned}$$

Let us copy this problem in the form

$$\Phi^+(\mu) - \Phi^-(\mu) = 0, \quad -1 < \mu < 1. \quad (6.10)$$



In the problem (6.10)  $\Phi^\pm(\mu)$  is the boundary values in interval  $-1 < \mu < 1$  of function

$$\Phi(z) = \lambda^+(z) \left[ M^+(z) + \frac{1}{2} E_\infty z + \frac{1}{2} E_0 \operatorname{ch} \frac{az_0}{\eta_0} \varphi(\eta_0, z) \right],$$

which is analytical function in complex plane with cut  $\mathbb{C} \setminus [-1, 1]$ .

The problem (6.10) is the problem special case about jump. It is problem of definition of analytical function on its jump on a contour  $L$ :

$$\Phi^+(\mu) - \Phi^-(\mu) = \varphi(\mu), \quad \mu \in L.$$

The solution of such problems in a class decreasing at infinitely remote point of functions is given by integral of Cauchy type

$$\Phi(z) = \frac{1}{2\pi i} \int_L \frac{\varphi(\tau) d\tau}{\tau - z}.$$

However, in the problem (6.10) unknown function  $\Phi(z)$  has at infinitely remote point following asymptotic

$$\Phi(z) = O(z), \quad z \rightarrow \infty.$$

Therefore (6.11) it is necessary to search for the solution of the problem in the class of the growing as  $z$  at vicinity of infinitely remote point.

According to [33] the general solution of the problem (6.10) is given by the formula  $\Phi(z) = C_1 z$ .

In explicit form the general solution of the problem (6.10) registers as follows:

$$\lambda(z) \left[ M(z) + \frac{1}{2} E_0 \operatorname{ch} \frac{az_0}{\eta_0} \varphi(\eta_0, z) + \frac{1}{2} E_\infty z \right] = C_1 z,$$

where  $C_1$  is the arbitrary constant.

From this general solution we can find function  $M(z)$

$$M(z) = -\frac{1}{2} E_\infty z - \frac{1}{2} E_0 \operatorname{ch} \frac{az_0}{\eta_0} \varphi(\eta_0, z) + \frac{C_1 z}{\lambda(z)}. \quad (6.11)$$

Let us remove the pole at the solution (6.11) at infinitely remote point. We receive, that

$$C_1 = \frac{1}{2} E_\infty \lambda_\infty. \quad (6.12)$$

## 8. COEFFICIENTS OF THE CONTINUOUS AND DISCRETE SPECTRA

Now we will eliminate polar singularity at the solution (6.11) at points  $\pm\eta_0$ . Let us allocate in the right part of the solution (6.11) members containing the polar singularity at points  $z = \eta_0$ . In the point vicinity  $z = \eta_0$  taking into account equality  $\lambda(\eta_0) = 0$  fairly following expansion

$$M(z) = -\frac{1}{2}E_\infty z - \frac{1}{2}E_0 \operatorname{ch} \frac{az_0}{\eta_0} \left[ \frac{\eta_0^2 - \eta_1^2}{\eta_0 - z} + \frac{\eta_0^2 + \eta_1^2}{\eta_0 + z} \right] + \frac{C_1 \eta_0}{\lambda'(\eta_0)(z - \eta_0) + \dots}.$$

From here is visible, that for pole elimination at the point  $z = \eta_0$  is necessary to equate to zero expression in a square bracket, calculated at  $z = \eta_0$ . Then we receive, that:

$$\operatorname{ch} \frac{az_0}{\eta_0} E_0 = -\frac{E_\infty \lambda_\infty \eta_0}{\lambda'(\eta_0)(\eta_1^2 - \eta_0^2)}. \quad (7.1)$$

The coefficient of continuous spectrum  $E(\eta)$  we will find from the formula (6.9)

$$\begin{aligned} E(\eta) \operatorname{ch} \frac{az_0}{\eta} &= \frac{1}{2\pi i} \cdot \frac{M^+(\eta) - M^-(\eta)}{\eta^2 - \eta_1^2} = \\ &= \frac{E_\infty \lambda_\infty}{2} \frac{\eta}{2\pi i(\eta^2 - \eta_1^2)} \left( \frac{1}{\lambda^+(\eta)} - \frac{1}{\lambda^-(\eta)} \right). \end{aligned} \quad (7.2)$$

Let us take advantage of the boundary condition for electric field

$$E_\infty + 2E_0 \operatorname{ch} \frac{az_0}{\eta_0} + 2 \int_{-1}^1 E(\eta) \operatorname{ch} \frac{az_0}{\eta} d\eta = e_s. \quad (7.3)$$

Let us substitute coefficient of the continuous spectrum (7.2) and coefficient of discrete spectrum (7.1) in the equation on electric field (7.3). We will receive the following equation

$$E_\infty - \frac{2E_\infty \lambda_\infty \eta_0}{\lambda'(\eta_0)(\eta_0^2 - \eta_1^2)} + E_\infty \lambda_\infty \frac{1}{2\pi i} \int_{-1}^1 \left( \frac{1}{\lambda^+(\eta)} - \frac{1}{\lambda^-(\eta)} \right) \frac{\eta d\eta}{\eta^2 - \eta_1^2} = e_s,$$

or

$$E_\infty \lambda_\infty \left[ \frac{1}{\lambda_\infty} - \frac{2\eta_0}{\lambda'(\eta)(\eta_0^2 - \eta_1^2)} + \frac{1}{2\pi i} \int_{-1}^1 \left( \frac{1}{\lambda^+(\eta)} - \frac{1}{\lambda^-(\eta)} \right) \frac{\eta d\eta}{\eta^2 - \eta_1^2} \right] = e_s.$$

Integral from this expression we will calculate by means of the contour integration and the theory of residual

$$\begin{aligned} & \int_{-1}^1 \left( \frac{1}{\lambda^+(\eta)} - \frac{1}{\lambda^-(\eta)} \right) \frac{\eta d\eta}{\eta^2 - \eta_1^2} = \\ & = \left[ \text{Res}_\infty + \text{Res}_{-\eta_1} + \text{Res}_{\eta_1} + \text{Res}_{\eta_0} + \text{Res}_{-\eta_0} \right] \frac{z}{\lambda(z)(z^2 - \eta_1^2)} = \\ & = -\frac{1}{\lambda_\infty} + \frac{1}{\lambda_1} + \frac{2\eta_0}{\lambda'(\eta)(\eta_0^2 - \eta_1^2)}. \end{aligned}$$

Substituting this integral in the previous equation, we find that

$$E_\infty = e_s \frac{\lambda_1}{\lambda_\infty}. \quad (7.4)$$

Now by means of a relation (7.1) we find

$$E_0 \text{ch} \frac{az_0}{\eta_0} = -e_s \frac{\lambda_1 \eta_0}{\lambda'(\eta_0)(\eta_0^2 - \eta_1^2)}. \quad (7.5)$$

By means of the found coefficients of discrete and continuous spectra we find an electric field profile in the slab

$$\begin{aligned} \frac{e(x)}{e_s} &= \frac{\lambda_1}{\lambda_\infty} - \frac{2\eta_0 \lambda_1}{\lambda'(\eta_0)(\eta_0^2 - \eta_1^2)} + \\ &+ \frac{\lambda_1}{2\pi i} \int_{-1}^1 \left[ \frac{1}{\lambda^+(\eta)} - \frac{1}{\lambda^-(\eta)} \right] \frac{\text{ch}(xz_0/\eta)}{\text{ch}(az_0/\eta)} \frac{\eta d\eta}{\eta^2 - \eta_1^2}. \end{aligned} \quad (7.6)$$

Integral from (7.6) we will calculate by means of residual and contour integration

$$\frac{1}{2\pi i} \int_{-1}^1 \left[ \frac{1}{\lambda^+(\eta)} - \frac{1}{\lambda^-(\eta)} \right] \frac{\text{ch}(xz_0/\eta)}{\text{ch}(az_0/\eta)} \frac{\eta d\eta}{\eta^2 - \eta_1^2} =$$

$$\begin{aligned}
&= \left[ \text{Res}_{\infty} + \text{Res}_{-\eta_1} + \text{Res}_{\eta_1} + \text{Res}_{\eta_0} + \text{Res}_{-\eta_0} + \sum_{k=-\infty}^{+\infty} \text{Res}_{t_k} \right] \frac{z \text{ch}(xz_0/z)}{\text{sh}(az_0/z)(z^2 - \eta_1^2)\lambda(z)} = \\
&= -\frac{1}{\lambda_{\infty}} + \frac{\text{ch}(xz_0/\eta_1)}{\lambda_1 \text{ch}(az_0/\eta_1)} + \frac{2\eta_0 \text{ch}(xz_0/\eta_0)}{\lambda'(\eta_0)(\eta_0^2 - \eta_1^2) \text{ch}(az_0/\eta_0)} + \\
&\quad + \frac{i}{az_0} \sum_{k=-\infty}^{+\infty} \frac{(-1)^k t_k^3 \text{ch}(xz_0/t_k)}{\lambda(t_k)(t_k^2 - \eta_1^2)}.
\end{aligned}$$

Here  $t_k$  are zeros of function  $\text{ch}(az_0/z)$ ,

$$t_k = \frac{2az_0 i}{\pi(2k+1)}, \quad k = 0, \pm 1, \pm 2, \dots,$$

also it has been considered that  $\text{sh}(az_0/t_k) = i(-1)^k$ .

Hence, an electric field profile in the slab of degenerate plasma is possible to express without quadratures

$$\frac{e(x)}{e_s} = \frac{\text{ch}(xz_0/\eta_1)}{\text{ch}(az_0/\eta_1)} - \frac{\lambda_1}{az_0} \sum_{k=-\infty}^{+\infty} \frac{t_k^3 \text{ch}(xz_0/t_k)}{\lambda(t_k)(t_k^2 - \eta_1^2) \text{sh}(az_0/t_k)},$$

or, that all the same,

$$\frac{e(x)}{e_s} = \frac{\text{ch}(xz_0/\eta_1)}{\text{ch}(az_0/\eta_1)} + \frac{i\lambda_1}{az_0} \sum_{k=-\infty}^{+\infty} \frac{(-1)^k t_k^3 \text{ch}(xz_0/t_k)}{\lambda(t_k)(t_k^2 - \eta_1^2)}. \quad (7.7)$$

Here  $\text{ch}(xz_0/t_k) = \text{ch} \frac{x}{a}(\pi k + \pi/2)$ .

Integrating expression (7.7) on  $x$  from  $-a$  for  $a$ , we receive

$$Q_1 = \int_{-a}^a e(x) dx = e_s \frac{2\eta_1}{z_0} \text{th} \frac{az_0}{\eta_1} - e_s \frac{2\lambda_1}{az_0^2} \sum_{k=-\infty}^{+\infty} \frac{t_k^4}{\lambda(t_k)(t_k^2 - \eta_1^2)}. \quad (7.8)$$

Thus, all the coefficients of the expansions (6.1) and (6.2) are found single-valued: the coefficient  $E_{\infty}$  is determined according to (7.4), the coefficient  $E_0$  is determined according to (7.5), the coefficient of the continuous spectrum  $E(\eta)$  is determined according to (7.2). Finding of the coefficients of discrete and continuous spectra of the expansions (6.1) and (6.2) completes the proving of existence of this expansions. Uniqueness of the solution in the form of expansions (6.1) and (6.2) can be proved easily with the use of contraposition method.

## 9. ENERGY ABSORPTION IN THE SLAB

The energy of an electromagnetic wave absorbed in the slab of degenerate plasma is calculated under the formula

$$Q = -\frac{\omega E_0}{8\pi} \int_{-a}^a \operatorname{Im} E(x') dx'. \quad (8.1)$$

Here dimensional quantities  $E(x)$ ,  $x$ ,  $a$  are written, and  $E(0) = E_0$ .

Let us pass in (8.1) to dimensionless quantities

$$e(x_1) = \frac{ev_F}{\nu \mathcal{E}_F} E(x), \quad x_1 = \frac{x}{l}, \quad e_s = \frac{ev_F}{\nu \mathcal{E}} E_0.$$

It is as a result received

$$Q = -\frac{\omega(\nu \mathcal{E}_F)^2 l e_s}{8\pi (ev_F)^2} \operatorname{Im} \int_{-a/l}^{a/l} e(x_1) dx_1. \quad (8.2)$$

Further we will designate

$$Q_1 = \int_{-a/l}^{a/l} e(x_1) dx_1.$$

Then according to (8.2) we have

$$Q = -\frac{\omega(\nu \mathcal{E}_F)^2 l e_s}{8\pi (ev_F)^2} \operatorname{Im} Q_1. \quad (8.2')$$

Besides, quantities  $x_1$  and  $a_1 = a/l$  we will designate again through  $x$  and  $a$ .

Earlier for electric field which we will present in the form

$$e(x) = E_\infty + 2E_0 \operatorname{ch} \frac{z_0 x}{\eta_0} + \int_{-1}^1 \operatorname{ch} \frac{z_0 x}{\eta} E(\eta) d\eta, \quad (8.3)$$

coefficients of continuous and discrete spectra have been calculated in the explicit form.

Integrating (8.3), we receive

$$Q_1 = 2aE_\infty + 4E_0 \frac{\eta_0}{z_0} \operatorname{sh} \frac{az_0}{\eta_0} + \frac{2}{z_0} \int_{-1}^1 \eta E(\eta) \operatorname{sh} \frac{az_0}{\eta} d\eta. \quad (8.4)$$

Substituting in (8.4) coefficients of discrete and continuous spectra, we receive the following expression for electric field

$$\begin{aligned} \frac{e(x)}{e_s} &= \frac{\lambda_1}{\lambda_\infty} - \frac{4\lambda_1\eta_0^2 \operatorname{th}(az_0/\eta_0)}{z_0\lambda'(\eta_0)(\eta_0^2 - \eta_1^2)} + \\ &+ \frac{2\lambda_1}{z_0} \frac{1}{2\pi i} \int_{-1}^1 \left[ \frac{1}{\lambda^+(\eta)} - \frac{1}{\lambda^-(\eta)} \right] \frac{\eta^2 \operatorname{th}(az_0/\eta)}{\eta^2 - \eta_1^2} d\eta. \end{aligned} \quad (8.5)$$

Integral from last expression we will calculate by means of the contour integration and the theory of residual

$$\begin{aligned} &\frac{1}{2\pi i} \int_{-1}^1 \left[ \frac{1}{\lambda^+(\eta)} - \frac{1}{\lambda^-(\eta)} \right] \frac{\eta^2 \operatorname{th}(az_0/\eta)}{\eta^2 - \eta_1^2} d\eta = \\ &= \left[ \operatorname{Res}_\infty + \operatorname{Res}_{-\eta_1} + \operatorname{Res}_{\eta_1} + \operatorname{Res}_{-\eta_0} + \operatorname{Res}_{\eta_0} + \sum_{k=-\infty}^{+\infty} \operatorname{Res}_{t_k} \right] \frac{z^2 \operatorname{th}(az_0/z)}{\lambda(z)(z^2 - \eta_1^2)} = \\ &= -\frac{az_0}{\lambda_\infty} + \frac{\eta_1}{\lambda_1} \operatorname{th} \frac{az_0}{\eta_1} + \frac{2\eta_0^2 \operatorname{th}(az_0/\eta_0)}{\lambda'(\eta_0)(\eta_0^2 - \eta_1^2)} - \frac{1}{az_0} \sum_{k=-\infty}^{+\infty} \frac{t_k^4}{\lambda(t_k)(t_k^2 - \eta_1^2)}, \end{aligned} \quad (8.6)$$

where points  $t_k$  are entered earlier.

Substituting (8.6) in (8.5), we receive

$$\frac{Q_1}{e_s} = \frac{2\eta_1}{z_0} \operatorname{th} \frac{az_0}{\eta_1} - \frac{2\lambda_1}{az_0^2} \sum_{k=-\infty}^{+\infty} \frac{t_k^4}{\lambda(t_k)(t_k^2 - \eta_1^2)}, \quad (8.7)$$

that is equivalent to expression (7.8).

## 10. CONCLUSION

In the present paper the linearized problem of plasma oscillations in slab (particularly, thin films) in external longitudinal alternating electric field is solved analytically. Specular boundary conditions of electron reflection

from the plasma boundary are considered. Coefficients of continuous and discrete spectra of the problem are found, and electron distribution function on the plasma boundary and electric field are expressed in explicit form.

Separation of variables leads to characteristic system of the equations. Its solution in space of the generalized functions allows to find the eigen solutions of initial system of equations of Boltzmann — Vlasov — Maxwell, correspond to continuous spectrum.

Then the discrete spectrum of this problem consisting of zero of the dispersion equations is investigated. Such zeros are three. One zero is infinite remote point of complex plane. It is correspond to the eigen solution "Drude mode" independent of boundary conditions. Others two points of discrete spectrum are differing with signs two zero of dispersion function. These zero are correspond to the eigen solution named "Debye mode".

It is found out, that on plane of parameters of a problem  $(\Omega, \varepsilon)$ , where  $\Omega = \omega/\omega_p, \varepsilon = \nu/\omega_p$ , there is such domain  $D^+$  (its exterior is domain  $D^-$ ), such, that if a point  $(\Omega, \varepsilon) \in D^-$  Debye mode is absent.

Under eigen solution of initial system its general solution is obtained. By means of boundary conditions in an explicit form expressions for coefficients of discrete and continuous spectra are found. Then in explicit form (without quadratures) absorption quantity of energy of electric field in slab of degenerate plasmas is found.

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